Parent Action Approach for the Duality between Non-Abelian Self-Dual and Yang-Mills-Chern-Simons Models.

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Abstract

It has been argued by some authors that the parent action approach cannot be used in order to establish the duality between the 2+1 Abelian and non-Abelian Self-Dual (SD) and Yang-Mills-Chern-Simons (YMCS) models for all the coupling regimes. We propose here an alternative (perturbative) point of view, and show that this equivalence can be achieved with the parent action approach.

There is a well-known duality between the (2+1)-dimensional Maxwell-Chern-Simons and Self-Dual [1] Abelian models; one can construct a so-called parent action [2] to show this result [1, 3, 4]. Here, we propose this same issue but viewed in an alternative way, which allows the generalization to the non-Abelian case. This is the main contribution of this letter.

The non-Abelian (NA) version of the so-called Self-Dual model [5] presents some well-known dificulties in order to establish the dual equivalence to the YMCS theory [4] for the full range of the coupling constant. The parent action approach first proposed by Deser and Jakiw [1] has proven to be useful in exhibiting the dual equivalence in the Abelian case; however, the situation is less understood in the non-Abelian case, where this equivalence has only been set up for the weak coupling regime [4]. In [3, 6] it is argued that the use of parent actions in this situation is ineffective to establish this duality since YMCS (or reciprocally SD) results to be dual to a non-local theory.

Recently, a technique claimed to be [7] alternative to the parent action approach has been shown to give the expected result for the Abelian case; then, it is inferred to work also for the non-Abelian case and for other cases too [7, 8, 9]. This method is based on the traditional idea of a local lifting of a global symmetry and may be realized by an iterative embedding of Noether counterterms. However also in this framework the parent action is lacking.

This is (briefly) the updated scenario for this problem. In this work, we proceed further and propose a novel way to solve the difficulties with the parent action suggested in Ref. [1], based on a perturbative analysis, and manifestly show the dual correspondence between the non-Abelian SD and YMCS models for the full range of the coupling constant, extending the proof proposed by Deser and Jackiw in the Abelian domain.

We shall show here that the parent action proposed in Ref. [1] actually interpolates YMCS with a (dual) theory, whose action is SD up to *fourth* order in the field.

The so-called Self-Dual Model [1, 5, 10] is given by the following action,

$$S_{SD}[f] = \int d^3x \left(\frac{\chi}{2}\, \epsilon_{\mu\nu\lambda}\, f^\mu\, \partial^\nu f^\lambda + \frac{m}{2}\, f_\mu f^\mu \right). \tag{1}$$

This actually is a self-dual model since the duality operation is, in 2+1-dimensions commonly defined by

$$^\star f_\mu = \frac{\chi}{m}\, \epsilon_{\mu\nu\lambda}\, \partial^\nu f^\lambda\,, \tag{2}$$

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Self or anti-self duality is dictated by $\chi = \pm 1$ and m is a constant to render the *-operation dimensionless. Here the Lorentz indices are represented by greek letters taking their usual values as $\mu, \nu, \lambda = 0, 1, 2$. The gauge invariant combination of a Chern-Simons term with a Maxwell action

$$S_{MCS}[A] = \int d^3x \left(\frac{1}{4m^2} F^{\mu\nu} F_{\mu\nu} - \frac{\chi}{2m}\, \epsilon^{\mu\nu\lambda}\, A_\mu\, \partial_\nu A_\lambda \right), \tag{3}$$

is the topologically massive theory, which is known to be equivalent [1] to the self-dual model (1). $F_{\mu\nu}$ is the usual Maxwell field strength,

$$F_{\mu\nu}[A] \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \; = 2\partial_{[\mu} A_{\nu]}. \tag{4}$$

This equivalence has been verified with the parent action approach [11]. We propose here an alternative way to generalizate it to the Non-Abelian case.

The non-Abelian version of the vector self-dual model (1), which is our main concern in this work, is given by

$$\mathcal{S}_{SD}[f] \equiv \int d^3x\, \frac{\chi}{2}\, \epsilon^{\mu\nu\lambda} \left(f_\mu{}^a \partial_\nu f_\lambda{}^a + \frac{\tau^{abc}}{3}\, f_\mu{}^a\, f_\nu{}^b\, f_\lambda{}^c \right) - \frac{m}{2}\, f_\mu{}^a f^{\mu a}, \tag{5}$$

where $f_\mu = f_\mu^a \tau^a$ is a vector field taking values in the Lie algebra of a symmetry group G and τ^a are the matrices representing the underlying non-Abelian gauge group with $a = 1, \ldots, \dim G$; τ_{abc} are the structure constants of the group [2].

The field-strength tensor is now defined as

$$F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,, A_\nu]\,, \tag{6}$$

and the covariant derivative is $D_\mu = \partial_\mu + [A_\mu\,, \,]$, where A_μ is also a vector field in the adjoint representation of the group G. This may be written with explicit group indices, using $\left[\tau^a, \tau^b\right] = \tau^{abc}\tau^c$; the field-strength reads as

$$F_{\mu\nu}{}^a[A] = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + \tau^{abc} A_\mu{}^b\; A_\nu{}^c\,. \tag{7}$$

Using the Parent action approach, the action (5) has been shown (in Ref. [4]) to be equivalent to the gauge invariant Yang-Mills-Chern-Simons (YMCS) theory

$$\mathcal{S}_{YMCS}[A] = \int d^3x\, tr \left[\frac{1}{2m}\, F^{\mu\nu a}[A]\, F_{\mu\nu}{}^a[A] - \chi\, \epsilon^{\mu\nu\lambda} \left(A_\lambda^a\, \partial_\mu A_\nu^a + \frac{\tau^{abc}}{3}\, A_\mu{}^a\, A_\nu{}^b\, A_\lambda^c \right) \right], \tag{8}$$

only in the weak coupling limit $m^{-1} \to 0$ so that the Yang-Mills term effectively vanishes [3]. In order to establish the dual equivalence between (5) and (8) for all coupling regimes, we write down the general parent action, which clearly contains the one for the Abelian case [1]:

$$\mathcal{S}_{Parent}[A, f] = \mathcal{S}_{CS}[A] - \int d^3x\, \left[\epsilon^{\mu\nu\lambda}\, F_{\nu\lambda}{}^a[A] f_\mu{}^a \, + m f_\mu{}^a f^{\mu a} \right], \tag{9}$$

where

$$\mathcal{S}_{CS}[A] \equiv \int d^3x\, \epsilon^{\mu\nu\lambda} \left(A_\mu^a \partial_\nu A_\lambda^a + \frac{\tau^{abc}}{3}\, A_\mu{}^a\, A_\nu{}^b\, A_\lambda^c \right). \tag{10}$$

First, we shall observe that the master action is invariant front the transformations; $A_\mu \to \Delta^{-1} A_\mu \Delta + \Delta_\mu, \quad f_\mu \to \Delta^{-1} f_\mu \Delta$. Where Δ_μ is a *pure gauge*: $\Delta_\mu := \Delta^{-1}\partial_\mu \Delta$, and Δ is a group element. We can verify this straightforwardly since the Chern-Simons term is known to be gauge invariant up boundary terms, and the coupling term depends on A only trough the field strength which is also gauge invariant.

[2] We assume that f_μ is in the adjoint representation.

[3] The coupling constant is given by the mass parameter, through $g^2 \equiv \frac{4\pi}{m}$

In fact, considering the redefinitions $A_\mu = \Delta^{-1}\tilde{A}_\mu\Delta + \Delta_\mu \quad , \, f_\mu = \Delta^{-1}\tilde{f}_\mu\Delta$, we get up to boundary terms,

$$\begin{aligned} \mathcal{S}_{Parent}[A,f] &\equiv \mathcal{S}_{CS}[\tilde{A}] - \int d^3x Tr \left(\epsilon^{\mu\nu\lambda}\,\Delta^{-1}\;F_{\nu\lambda}[\tilde{A}]\;\Delta\;f_\mu + m\;f_\mu f^\mu\right) \\ &= \mathcal{S}_{CS}[\tilde{A}] - \int d^3x Tr \left(\epsilon^{\mu\nu\lambda}\,\Delta^{-1}F_{\nu\lambda}[\tilde{A}]\tilde{f}_\mu\Delta + m\,\Delta^{-1}\tilde{f}_\mu\tilde{f}^\mu\Delta^{-1}\right). \end{aligned} \tag{11}$$

Therefore,

$$\mathcal{S}_{Parent}[A,f] \equiv \mathcal{S}_{Parent}[\tilde{A},\tilde{f}]. \tag{12}$$

Varying $\mathcal{S}_{Parent}$ with respect to f, we obtain

$$f^{\mu a} = -\frac{1}{2m}\epsilon^{\mu\nu\lambda}\,F_{\nu\lambda}{}^a[A]; \tag{13}$$

plugging this back into (9), and using

$$\epsilon^{\mu\nu\alpha}\epsilon_{\mu\nu\lambda} = 2\,\delta^\alpha_\lambda, \tag{14}$$

we recover the YMCS-action, Eq. (8).

Now, following strictly the standard program of the master action approach [2], we must vary the master action with respect to A, and use the resulting equation to solve A in terms of the other field, f. Finally, one shall eliminate A from the action.

Now, we vary with respect to A and obtain:

$$2\epsilon^{\mu\nu\lambda}\,[\partial_{[\nu}A^{\;a}_{\lambda]} + \tau^{abc}\,A_\nu{}^b\,A_\lambda{}^c - 2\partial_{[\nu}f_{\lambda]}{}^a - 2\tau^{abc}\,A_\nu{}^b\,f_\lambda{}^c] = 0, \tag{15}$$

by using (14), one can eliminate the Levi-Civita symbol, and from (7) we can rewrite this equation as

$$F_{\nu\lambda}{}^a[A_\mu - f_\mu] = \tau^{abc}\,f_\nu{}^b\,f_\lambda{}^c. \tag{16}$$

In the Abelian case this is

$$F_{\nu\lambda}[A_\mu - f_\mu] = 0, \tag{17}$$

then we have

$$A_\mu = f_\mu + \Delta_\mu. \tag{18}$$

Putting this back into the action (9) , we recover the SD theory (1) up to boundary terms.

The solution to the general equation (Non-Abelian) (16) is less understood; this is the origin for the difficulties for establishing duality with the SD-model.

In the non-Abelian case, the field strength does not determine the gauge potentials $(A_\mu - f_\mu)$ up to gauge transformations; this is known as the Wu-Yang ambiguity [12]. In other words, the operator F cannot be inverted in equation (16) in a unique way [4].

In Ref. [3], a solution is found by using the Fock-Schwinger gauge, yielding a (second order in f) non-local solution.

We propose an alternative way to see this and tackle this problem. Let us recall that one must find a *functional* solution, $A_\mu = A_\mu[f_\nu]$ of this equation and replace it into the action (9), which will result expressed in terms of f. However, one can assume that a solution exists in this way at least perturbatively.

Lets assume a formal development of this functional in the form :

$$A_\mu = A^{(0)}_\mu + A^{(1)}_\mu[f_\nu] + A^{(2)}_\mu[f_\nu] + ... \tag{19}$$

where $A^{(0)}_\mu$ is independent of f_μ, $A^{(1)}_\mu$ is first order in f_μ, thus this shall be a linear functional (it may be a no-local operator) of f_μ, $A^{(2)}_\mu$ is second order in f_μ and so on.

[4] In the Fock-Schwinger gauge, the (non-local) solution of (16) is $A_\lambda{}^a = f_\lambda{}^a + \int_0^1 dt\,t\,x^\nu(\tau^{abc}\,f_\nu{}^b\,f_\lambda{}^c)|_{tx^\mu}$, where x^μ is the space-time point [13]. Notice that the non-local part, is second order in f.

Actually, we admit that the functional $A_\mu[f_\nu]$ is analytical enough in order to admit this development (at least to first order). One can perform a perturbative analisys of the solutuion and solve this order by order.

Putting this development into equation (9), and assuming that this is satisfied to each order, we obtain two equations for the zeroth and first orders respectively; the zeroth order is:

$$\epsilon^{\mu\nu\lambda}\, F_{\nu\lambda}[A^{(0)}_\mu] = 0. \tag{20}$$

Thus, using once more (**??**), this reads

$$F_{\nu\lambda}[A^{(0)}_\mu] = 0. \tag{21}$$

This implies that the zeroth order corresponds to a pure gauge which does not contribute to the action (9). So, $A^{(0)}_\mu$ can be dropped out of the solution (19) [5].

The first order equation reads,

$$\partial_{[\nu}\left(A^{(1)\ a}_{\lambda]} - f^{\ a}_{\lambda]}\right) + \tau^{abc}\left(A^{(1)\ b}_{\nu} - f^{\ b}\right) A^{(0)\ c}_{\lambda} = 0. \tag{22}$$

Since we are interested in obtaining the self-dual model whose action is third order in the potential field, let us substitute the perturbative solution (19) into the master action and keep terms of third order in f,

$$\mathcal{S} = \mathcal{S}_{Parent}[A^{(1)}, f] + \epsilon^{\mu\nu\lambda}\, A^{(2)}_\mu \partial_{[\nu} A^{(1)}_{\lambda]} + \epsilon^{\mu\nu\lambda}\, A^{(1)}_\mu \partial_\nu A^{(2)}_\lambda - 2\epsilon^{\mu\nu\lambda}\, f_\mu \partial_{[\nu} A^{(2)}_{\lambda]} + o^4(f). \tag{23}$$

Integrating out by parts, we obtain:

$$\mathcal{S} = \mathcal{S}_{Parent}[A^{(1)},\, f] + \epsilon^{\mu\nu\lambda}\, 2[A^{(1)}_\mu - f_\mu]\partial_{[\nu} A^{(2)}_{\lambda]} + o^4(f) \tag{24}$$

Then, we can make a crucial observation in order to find the dual action: *only the first and second orders contribute to a dual third order action.*

Let us calculate a solution for the first order. Below, we shall prove that the second order will not be actually needed.

Like in the Abelian case, we can see that

$$A^{(1)}_\mu = f_\mu\,, \tag{25}$$

and $A^{(0)}_\mu = \Delta_\mu$ is a solution to (21) and (22) [6].

Thus, using the fact discussed above, that a pure gauge is irrelevant for the action, we can write

$$A_\mu[f] = f_\mu + A^{(2)}_\mu[f]. \tag{26}$$

Substituting this into (24), we can see that the second term identically vanishes and a second order, therefore, $A^{(2)}[f]$ will contribute to the action only in its *fourth* order. Finally, we get:

$$\begin{aligned} \mathcal{S}[f] &= \mathcal{S}_{Parent}[f_\mu + A^{(2)}_\mu[f]\,,f_\mu] + o^4(f) \\ &= -\epsilon^{\mu\nu\lambda}\left[f_\mu^{\ a}\partial_\nu f_\lambda^{\ a} + \frac{2\tau^{abc}}{3}\, f_\mu^{\ a}\, f_\nu^{\ b}\, f_\lambda^{\ c}\right] - m\, f_\mu^{\ a} f^{\mu a} + o^4(f). \end{aligned} \tag{27}$$

We may rescale $f_\mu \rightarrow \frac{1}{2} f_\mu$ and recover the SD-theory,

[5] As it has been shown above, one can redefine $(A_\mu\,, f_\mu) \rightarrow (\Delta^{-1}\tilde{A}_\mu\Delta + \Delta_\mu\,, \Delta^{-1}\tilde{f}_\mu\Delta)$, to obtain an equivalent parent action. Note also that these transformations do not change the order (in f) of the expressions.

[6] Notice that (16) is equivalent to $F(A - f) = 0$ up to second order. This implies that the difference $A - f$, up to second order, is a pure gauge. Thus, we may conclude that solution (25) is essentially unique.

$$\mathcal{S}[f] = -\frac{1}{4}\epsilon^{\mu\nu\lambda}\left[f_\mu{}^a \partial_\nu f_\lambda{}^a + \frac{\tau^{abc}}{3} f_\mu{}^a f_\nu{}^b f_\lambda{}^c \right] - (\frac{m}{4}) f_\mu{}^a f^{\mu a} + o^4(f). \tag{28}$$

This completes the proof of our main statement.

One may conclude that YMCS is (dual) equivalent to a theory (described by the field f) which coincides with the Self-Dual model for an arbitrary coupling constant, m^{-1}, up to fourth order in f. The well-known non-local contributions would appear at higher orders in f.

This result has useful consequences for the bosonization identities between the massive Thirring model and the topologically massive model, whenever the fermions carry non-Abelian charges [8, 11] .

Here, our strategy was somewhat different with respect to the usual analysis. We have tackled this problem from a perturbative point of view; this may be helpful to solve similar problems and to establish other dual equivalences between models, besides the additional advantage of rendering more straightforward the treatment of non-Abelian mathematical structures.

Aknowledgements: The author is indebted to Prof. C. Wotzasek for pointing out the relevance of the problem and Prof. J. A. Helayel-Neto for invaluable discussions and pertinent corrections on the manuscript. Thanks are due to the GFT-UCP for the kind hospitality. CNPq is also acknowledged for the invaluable financial help.

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